Math 451: Introduction to General Topology
Lecture 18

Def. Let X, Y be top. spaces. Call a function $f: X \to Y$ a *homeomorphism* if it is a bijection and both f and f^{-1} are continuous; equiv. f is a bijection and maps open sets to open sets and back, i.e. f^{-1} also maps open sets to open sets. Call X and Y *homeomorphic* if there exists a homeomorphism $X \to Y$; in other words, up to relabeling the points, X and Y are the same top. spaces.

Examples. (a) $2^{\mathbb{N}}$ and the Cantor set $C \subseteq [0,1]$ are homeomorphic.
(b) The Baire space $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$. (optional HW)
(c) The profinite top on $\mathbb{Z}$ is a ctbl Hausdorff 0-dim metrizable space, so it is homeomorphic to $\mathbb{Q}$, by a theorem of Sierpinski.

Def. Let X Y be top. spaces. Call a function $f: X \to Y$ an *embedding* if it is a homeomorphism from X to its image $f(X)$; equiv. f is injective, continuous, and maps open sets of X to open subsets of $f(X)$ in the subspace top of $f(X)$ (not necessarily open in Y).

Examples. (a) Let $f: 2^{\mathbb{N}} \hookrightarrow \mathbb{R}$ be the homeomorphism from $2^{\mathbb{N}}$ to C, the Cantor set. Then f is an embedding of $2^{\mathbb{N}}$ into $\mathbb{R}$, but $f(2^{\mathbb{N}}) = C$ is not in $\mathbb{R}$.
(b) For any top. space X and $Y \subseteq X$, the inclusion map $i: Y \hookrightarrow X : y \mapsto y$ is an embedding.

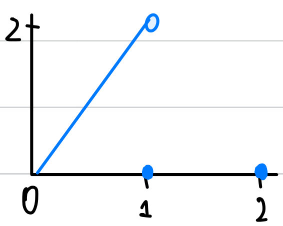
Recall. A continuous injection is typically not an embedding, e.g. let $X := \mathbb{R}$ with the discrete top and let $Y := \mathbb{R}$ with the usual (Euclidean) top, then the identity map $id: X \to Y : x \mapsto x$ is a continuous injection, but it's not an embedding because $\{0\}$ is open in X but its image $\{0\}$ is not open in $id(X) = Y$.

Limits of functions.

Def. Let X, Y be top. spaces, $f: X \to Y$, and $x_0 \in X$. We say a point $y_0 \in Y$ is a *limit of f at x_0*, and write $\lim\limits_{x \to x_0} f(x) = y_0$, if for every open $V \ni y_0$, $f^{-1}(V) \cup \{x_0\}$ is a neighbourhood of x_0.

Obs. If $x_0 \in X$ is a isolated in X (i.e. $\{x_0\}$ is open), then $\lim\limits_{x \to x_0} f(x) = y_0$ for all $y_0 \in Y$. This is why limits of functions are only useful at nonisolated points.

Example. Let $X := [0,1] \cup \{2\} \subseteq \mathbb{R}$ with the subspace top. Define $f: X \to \mathbb{R}$ by



$$f(x) := \begin{cases} 2x & \text{if } x < 1. \\ 0 & \text{if } x = 1 \\ 0 & \text{if } x = 2 \end{cases}$$

Then $\lim\limits_{x \to 1} f(x) = 2$ because for any open ball $(2-\varepsilon, 2+\varepsilon) \ni 2$, we have $f^{-1}(2-\varepsilon, 2+\varepsilon) = (1-\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}) \setminus \{1\}$ if $0 < \varepsilon < 2$ because $f(1) \notin (2-\varepsilon, 2+\varepsilon)$ so $f^{-1}(2-\varepsilon, 2+\varepsilon) \cup \{1\} = (1-\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2})$ which is a neighb. of 1.

Also, $\lim\limits_{x \to 2} f(x) = -7000000$ (or any other number) because 2 is isolated in X.

Prop. Let X, Y be top. spaces and $f: X \to Y$. Let $x_0 \in X$ be a nonisolated point. If Y is Hausdorff then $\lim\limits_{x \to x_0} f(x)$ is unique if exists.

Proof. HW

Prop. Let X, Y be top. spaces, $f: X \to Y$, $x_0 \in X$, and $y_0 \in Y$.

(1) $\lim\limits_{x \to x_0} f(x) = y_0$

$\Downarrow$ $\quad \Uparrow$ for 1^{st} cftbl X

(2) $\lim\limits_{n \to \infty} f(x_n) = y_0$ for every sequence $(x_n) \subseteq X \setminus \{x_0\}$ that converges to x_0.

Proof. (1)$\Rightarrow$(2). Suppose $\lim\limits_{x \to x_0} f(x) = y_0$ and let $(x_n) \subseteq X \setminus \{x_0\}$ be a sequence converging to x_0. Fix an open $V \ni y_0$. Then $f^{-1}(V) \cup \{x_0\}$ is a neighbourhood of x_0, so because $x_n \to x_0$, we have $\forall^\infty_n\ x_n \in f^{-1}(V) \cup \{x_0\}$. Since $x_n \neq x_0\ \forall n \in \mathbb{N}$, this implies $\forall^\infty_n\ x_n \in f^{-1}(V)$, i.e. $\forall^\infty_n\ f(x_n) \in V$, which means that $f(x_n) \to y_0$ as $n \to \infty$.

(2)⇒(1). Suppose X is 1^{st} cfbl and show not (1) ⇒ not (2). Assume $\lim_{x \to x_0} f(x) = y_0$ doesn't hold, i.e. $\exists$ open $V \ni y_0$ s.t. $f^{-1}(V) \cup \{x_0\}$ is not a neighbourhood of x_0. Fix a cfbl neighbourhood basis (U_n) for x_0, so $U_n \not\subseteq f^{-1}(V) \cup \{x_0\}$ for each $n \in \mathbb{N}$. WLOG, assume (U_n) is decreasing. By AC, get $x_n \in U_n \setminus (f^{-1}(V) \cup \{x_0\})$. Then the sequence $(x_n) \subseteq X \setminus \{x_0\}$ converges to x_0 because $\forall U_n\ \forall m \geq n\ x_m \in U_m \subseteq U_n$. However, because $x_n \notin f^{-1}(V)$, we have $f(x_n) \notin V$ for all $n \in \mathbb{N}$, so $(f(x_n))$ doesn't converge to y_0. $\square$

Obs. For top. spaces X, Y and an isolated $x_0 \in X$, every $f: X \to Y$ is continuous at x_0 (because for any $V \ni f(x_0)$, $f^{-1}(V) \supseteq \{x_0\}$ hence is a neighbourhood of x_0).

Continuity via limits. Let X, Y be top. spaces, $f: X \to Y$, and $x_0 \in X$.

(1) f is continuous at x_0.

⇕

(2) $\lim_{x \to x_0} f(x) = f(x_0)$.

⇓⇑ for 1^{st} ctbl X

(3) $\lim_{n \to \infty} f(x_n) = f(x_0)$ for all $(x_n) \subseteq X \setminus \{x_0\}$ converging to x_0.

Proof. HW

Topology generated by functions. Let X be a set and Y be a top. space. Let $\mathcal{F}$ be a set of functions $X \to Y$. The *topology on X generated by $\mathcal{F}$* is the smallest (least collection of open sets) which makes all functions in $\mathcal{F}$ continuous, namely, it is the top $\mathcal{T}_{\mathcal{F}}$ generated by the sets of the form $f^{-1}(V)$ where $f \in \mathcal{F}$ and $V \subseteq Y$ is open. Then a basis for this topology is formed by the finite intersections of these preimages, i.e. the sets $[f_1 \mapsto V_1, f_2 \mapsto V_2, \ldots, f_n \mapsto V_n] := \bigcap_{i=1}^n f_i^{-1}(V_i)$, where $f_i \in \mathcal{F}$, $V_i \subseteq Y$ are open.

We will use this construction even when X is equipped with some natural top.

Examples. Let $X := \mathbb{R}$ and $Y := \mathbb{R}$.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, for example, $f(x) := |x|$. Then the top on $\mathbb{R}$ generated by f is $\mathcal{T}_f := \{ f^{-1}(V) : V \subseteq \mathbb{R} \text{ open} \}$. This is a weaker top than the Euclidean top on $X := \mathbb{R}$ becase f is cont. wrt the Euclid. top, so $f^{-1}(V)$ are Euclid-open. $\mathcal{T}_f$ is not necessarily strictly weaker (e.g. if $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, e.g. the identity map), but $\mathcal{T}_f$ is strictly weaker for $f(x) = |x|$, becase if $V \subseteq [0, \infty)$ open then $f^{-1}(V) = V \cup (-V)$. In particular, $+r$ and $-r$ are inseparable for all $r > 0$, hence $\mathcal{T}_f$ is not Hausdorff.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a discontinuous function, so $f^{-1}(V)$ is not Euclid-open for some open $V \subseteq \mathbb{R}$. Then $\mathcal{T}_f$ is not weaker than Euclidean top, neither it is stronger.